미분기하 문제풀이 2차

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#4.17

α: unit speed curve which is C^∞ on (a,b).

Define the curve $\beta : (a,b) \to \mathbb{R}^3$

by $\beta(s) = \alpha'(s) = \vec{T}(s)$.

Then the arc length of β is

$$\bar{s} = \int_0^s \|\beta'(\sigma)\| d\sigma$$

$$= \int_0^s \|\vec{T}'(\sigma)\| d\sigma.$$

a) $\dfrac{d\bar{s}}{ds} = \|T'(s)\| = \kappa(s)$

b) β is regular iff $\beta'(s) \neq \vec{0}$

or $\|\vec{T}'(s)\| \neq 0$

everywhere on (a,b)

i.e. β is regular iff $\kappa(s) > 0$ everywhere.

c) Let γ be the curve on (a,b)

defined by $\gamma(s) = \vec{N}(s)$.

Then $s^* = \int_0^s \|\vec{N}'(\sigma)\| d\sigma = \int_0^s \sqrt{\kappa^2 + \tau^2}\, d\sigma$

$= \sqrt{\cancel{\kappa^2+\tau^2}}\, s$.

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4.20 Suppose $\exists$ a constant unit vector $\vec{n}$ s.t. $\langle \vec{T}(s), \vec{n} \rangle = 0$ for $\forall s \in (a, b)$. show that $\alpha(s)$ lies in a plane.

pf) $\langle \alpha'(s), \vec{n} \rangle = 0$ for $\forall s \in (a, b)$

$$\frac{d}{ds}\langle \alpha(s), \vec{n} \rangle = \langle \alpha'(s), \vec{n} \rangle + \langle \alpha(s), \vec{n}' \rangle$$

$$= \langle \alpha'(s), \vec{n} \rangle + \langle \alpha(s), \vec{0} \rangle$$

$$= 0 + 0 = 0$$

Therefor $\langle \alpha(s), \vec{n} \rangle \equiv$ constant

$\langle \alpha(s), \vec{n} \rangle = \langle \alpha(s_0), \vec{n} \rangle$ for $\forall s$

$\langle \alpha(s) - \alpha(s_0), \vec{n} \rangle = 0$ for $\forall s$

i.e. $\alpha(s)$ lies in a plane through $\alpha(s_0)$ with a unit normal.

p.41

#4.40

a) $\alpha(s)$: unit speed plane curve which is C^∞.

$\beta(s)$: its involute.

$\Rightarrow$ $\alpha(s)$ & $\beta(s)$ lies in the same plane.

pf) Let $\alpha(s)$ be a unit speed curve of class C^∞ which lies in the plane $\{x \mid \langle x - x_0, \vec{n} \rangle = 0\}$, and $\beta(s)$ its involute.

since $\beta(s)$ is an involute of $\alpha(s)$,

$\beta(s) = \alpha(s) + \lambda(s)\alpha'(s)$ for some function λ of s.

$$\begin{aligned}\langle \beta(s) - x_0, \vec{n} \rangle &= \langle \alpha(s) - x_0 + \lambda(s)\alpha'(s), \vec{n} \rangle \\ &= \underline{\langle \alpha(s) - x_0, \vec{n} \rangle}^{=0} \\ &\quad + \lambda(s) \underbrace{\langle \alpha'(s), \vec{n} \rangle}_{=0}\end{aligned}$$

Since $\alpha'(s)$ is a vector in the plane, $\langle \alpha'(s), \vec{n} \rangle = 0$, so $\langle \beta(s) - x_0, \vec{n} \rangle = 0$.

p.41

#4.40

b) $\alpha(s)$: a helix

$\beta(s)$: its involute $\rightarrow$ $\beta(s)$: plane curve

pf) $\beta(s) = \alpha(s) + \lambda(s)\alpha'(s)$

$\beta'(s) = \alpha'(s) + \lambda'(s)\alpha'(s) + \lambda(s)\alpha''(s)$

Since $\langle \beta'(s), \alpha'(s) \rangle = 0$, $\langle \alpha'(s), \alpha''(s) \rangle = 0$,

$$
\begin{aligned}
0 &= \langle \beta'(s), \alpha'(s) \rangle \\
&= \langle \alpha' + \lambda'\alpha' + \lambda\alpha'', \alpha' \rangle \\
&= (1+\lambda')\langle \alpha', \alpha' \rangle + \lambda \langle \alpha'', \alpha' \rangle \\
&= 1+\lambda' \quad \therefore \lambda' = -1 \quad \& \ \lambda(s) = -s + c
\end{aligned}
$$

$\therefore$ $\beta(s)$ should be of ~~the~~ form

$\beta(s) = \alpha(s) + (c-s)\alpha'(s)$.

$\exists$ a unit constant vector $\vec{u}$ s.t.

$\langle \alpha', \vec{u} \rangle \equiv$ constant.

Therefore $\langle \alpha'', \vec{u} \rangle \equiv 0$.

$$
\begin{aligned}
\langle \beta'(s), \vec{u} \rangle &= \langle \alpha'(s) - \alpha'(s) + (c-s)\alpha'', \vec{u} \rangle \\
&= 0
\end{aligned}
$$

Let $x_0 = \beta(s_0)$. Then

$$
\begin{aligned}
\langle \beta(s) - x_0, \vec{u} \rangle &= \langle \beta(s_0), \vec{u} \rangle - \langle \beta(s_0), \vec{u} \rangle \\
&= 0 \quad \text{for } \forall s.
\end{aligned}
$$

#5.2 Solve the system of linear differential equations

$$\vec{u_1}' = \frac{1}{1+s^2}\vec{u_2}$$

$$\vec{u_2}' = -\frac{1}{1+s^2}\vec{u_1}$$

Let $\vec{u_1} = (x(s), y(s), 0)$, $\vec{u_2} = (z(s), w(s), 0)$.

Then find out x, y, z, w satisfying

$$x' = \frac{1}{1+s^2}z$$

$$y' = \frac{1}{1+s^2}w$$

$$z' = -\frac{1}{1+s^2}x$$

$$w' = -\frac{1}{1+s^2}y.$$

Then

$$\vec{u_1} = \left(\cos(\tan^{-1}(s)), \sin(\tan^{-1}(s)), 0\right)$$

$$\vec{u_2} = \left(-\sin(\tan^{-1}(s)), \cos(\tan^{-1}(s)), 0\right).$$

Put

$$\alpha(s) = \int_0^s \vec{u_1}(\sigma)\,d\sigma$$

$$= \left(\int_0^s \cos(\tan^{-1}(\sigma))\,d\sigma, \int_0^s \sin(\tan^{-1}\sigma)\,d\sigma, 0\right)$$

p.45

#5.3

a) $\kappa = c$ $(\neq 0)$ $\tau = 0$ ∴ plane unit speed curve.

We know that a circle with radius $\frac{1}{c}$ has a curvature c, and torsion $\tau = 0$. By thm. 5.2, it is unique (up to a position) if the initial condition is given.

Hence any plane curve with constant curvature must be a circle.

b) Let $\alpha(s)$ be a spherical unit speed curve with constant curvature $\kappa = c > 0$. Then

i) $\langle \alpha'(s), \alpha'(s) \rangle \equiv 1$

ii) $\exists \vec{m}$) $\langle \alpha(s) - \vec{m}, \alpha(s) - \vec{m} \rangle = r^2$ $(r > 0)$

iii) $\|\alpha''(s)\| = c$.

$0 = \langle \alpha', \alpha(s) - \vec{m} \rangle \Rightarrow \langle \alpha'', \alpha(s) - \vec{m} \rangle + \langle \alpha', \alpha' \rangle = 0$

$\therefore \langle \alpha'', \alpha(s) - \vec{m} \rangle = -1$

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#5.3

i.e. $\langle \kappa \vec{N}, \alpha(s) - \vec{m} \rangle = -1$

$$\kappa \langle \vec{N}, \alpha(s) - \vec{m} \rangle = -1$$

$$\langle \vec{N}', \alpha(s) - \vec{m} \rangle + \langle \vec{N}, \overset{\vec{T}}{\alpha'(s)} \rangle = 0$$

$$\langle -\kappa \vec{T} + \tau \vec{N}, \alpha(s) - \vec{m} \rangle = 0$$

$$0 = \langle -\kappa \vec{T}, \alpha(s) - \vec{m} \rangle + \tau \langle \vec{N}, \alpha(s) - \vec{m} \rangle$$

$$= -\kappa \langle \alpha', \alpha(s) - \vec{m} \rangle + \tau \langle \frac{1}{\kappa} \alpha'', \alpha(s) - \vec{m} \rangle$$

$$= -\kappa \underline{\langle \alpha', \alpha(s) - \vec{m} \rangle} + \frac{\tau}{\kappa} \langle \alpha'', \alpha(s) - \vec{m} \rangle$$

$$= -\kappa \cdot 0 + \frac{\tau}{\kappa} \cdot (-1)$$

$\therefore \boxed{\tau \equiv 0}$ i.e. $\alpha(s)$ lies in a plane. Therefore, it must be a circle. (In fact, it must lie in an intersection of a sphere & a plane.)

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#5.3

i.e. $\langle \kappa \vec{N}, \alpha(s) - \vec{m} \rangle = -1$

$\kappa \langle \vec{N}, \alpha(s) - \vec{m} \rangle = -1$

$\langle \vec{N}', \alpha(s) - \vec{m} \rangle + \langle \vec{N}, \overset{\vec{T}}{\alpha'(s)} \rangle = 0$

$\langle -\kappa \vec{T} + \tau \vec{N}, \alpha(s) - \vec{m} \rangle = 0$

$0 = \langle -\kappa \vec{T}, \alpha(s) - \vec{m} \rangle + \tau \langle \vec{N}, \alpha(s) - \vec{m} \rangle$

$= -\kappa \langle \alpha', \alpha(s) - \vec{m} \rangle + \tau \langle \frac{1}{\kappa} \alpha'', \alpha(s) - \vec{m} \rangle$

$= -\kappa \underline{\langle \alpha', \alpha(s) - \vec{m} \rangle} + \frac{\tau}{\kappa} \langle \alpha'', \alpha(s) - \vec{m} \rangle$

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$\therefore \boxed{\tau \equiv 0}$ i.e. $\alpha(s)$ lies in a plane. Therefore, it must be a circle. (In fact, it must lie in an intersection of a sphere & a plane.)