Midterm Examination for topology (31, Oct., 2019)

1. Show that a subset A of a topological space X is closed if and only if A contains all its limit points.

2. Show that every separable metric space is second countable.

3. Let X be a space and $\{A_\alpha : \alpha \in I\}$ a family of connected subset of X for which $\cap_{\alpha\in I} A_\alpha$ is not empty. Show that $\cup_{\alpha\in I} A_\alpha$ is connected.

4. Let $C = \{(t,0) \in \mathbb{R}^2 \mid 0 \le t \le 1\} \cup \{(\frac{1}{n}, t) \in \mathbb{R}^2 \mid n \in \mathbb{N}, 0 \le t \le 1\} \cup \{(0,1) \in \mathbb{R}^2\}$ be a subspace of $\mathbb{R}^2$.

   (a) Is C connected?

   (b) Is C locally connected?

   (c) Is C path-connected?

5. Let X be a connected and locally path connected space.

   (a) Is every path component of an open subset O of X open?

   (b) Is X path connected?

Solution Set of Midterm Examination for topology (31, Oct., 2019)

1. Let A be a closed subset of X and x a limit point of A. If $x \notin A$, in other words, x lies in the open set $X \setminus A$, there is an open set U such that $x \in U \subset X \setminus A$. Since U does not contain element of A, x cannot be a limit point of A, a contradiction. Hence $x \in A$ or $A' \subset A$.

   Suppose A contains all its limit points. Let $x \in X \setminus A$. Since x is not a limit point of A, there is an open set U about x such that $U - \{x\} \cap A = \emptyset$.. Therefore $x \in U \subset X \setminus A$. This means $X \setminus A$ is open or A is closed in X.

2. See theorem 4.7.

3. See Theorem 5.5.

4. Let $C = \{(t, 0) \in \mathbb{R}^2 \mid 0 \le t \le 1\} \cup \{(\frac{1}{n}, t) \in \mathbb{R}^2 \mid n \in \mathbb{N}, 0 \le t \le 1\} \cup \{(0, 1) \in \mathbb{R}^2\}$ be a subspace of $\mathbb{R}^2$.

   (a) Let $K = \{(t, 0) \in \mathbb{R}^2 \mid 0 \le t \le 1\} \cup \{(\frac{1}{n}, t) \in \mathbb{R}^2 \mid n \in \mathbb{N}, 0 \le t \le 1\}$. Since K is path connected, K is connected. Note that $K \subset C \subset \bar{K}$ and $\bar{K} = \{(t, 0) \in \mathbb{R}^2 \mid 0 \le t \le 1\} \cup \{(\frac{1}{n}, t) \in \mathbb{R}^2 \mid n \in \mathbb{N}, 0 \le t \le 1\} \cup \{(0, t) \in \mathbb{R}^2 \| 0 \le t \le 1\}$, which is path connected and so connected. Hence C is connected by Corollary of Theorem 5.4.

   (b) Want to show C in not locally connected at $(0, 1)$. Consider the open ball $B((0, 1), \frac{1}{2})$ containing $(0, 1)$. For any open set U about $(0, 1)$ which is contained in the open ball, there is $\epsilon > 0$ such that $B((0, 1), \epsilon) \subset U$ there is n such that $(\frac{1}{n}, 1) \in C \cap B((0, 1), \epsilon)$. Choose $r > 0$ so that $\frac{1}{n+1} < r < \frac{1}{n}$. Consider the two disjoint open subsets $(-\infty, r) \times \mathbb{R}$ and $(r, \infty) \times \mathbb{R}$ of $\mathbb{R}^2$. The intersections of U and these two open sets makes up the separation of U, which means U cannot be connected.

   (c) Suppose that$p : [0, 1] \longrightarrow C$ is a path starting from $(0, 1)$. We assert that the set $p^{-1}(\{(0, 1)\})$ is both open and closed in $[0, 1]$, and therefore $p^{-1}(\{(0, 1)\}) = [0, 1]$ by the connectivity of $[0, 1]$.

   Clearly $p^{-1}(\{(0, 1)\})$ is closed since p is continuous.

   Let's show $p^{-1}(\{(0, 1)\})$ is open. Let $B((0, 1), \epsilon)$ be an open ball of $(0, 1)$ which does not intersect the x-axis. Given an arbitrary point

x_o of $p^{-1}(\{(0,1)\})$, we can choose an open ball $B(x_o,\delta)$ such that $p(B(x_o,\delta)) \subset B((0,1),\epsilon)$. Since $B(x_o,\delta)$ is connected, $p(B(x_o,\delta))$ is connected. If $(\frac{1}{n},t_o) \in B((0,1),\epsilon)$, choose $r>0$ so that $\frac{1}{n+1} < r < \frac{1}{n}$. Consider the two disjoint open subsets $(-\infty,r)\times\mathbb{R}$ and $(r,\infty)\times\mathbb{R}$ of $\mathbb{R}^2$. Because $p(B(x_o,\delta))$ lies in $B((0,1)),\epsilon) \subset C$ and does not touch the x-axis, $p(B(x_o,\delta))$ does not intersect the line $x=r$. Therefore, it lies in the union of the sets $(-\infty,r)\times\mathbb{R}$ and $(r,\infty)\times\mathbb{R}$. Since $p(B(x_o,\delta))$ is connected, and $(0,1)$ is contained of the first set, we know that $p(B(x_o,\delta)) \subset (-\infty,r)\times\mathbb{R}$. Hence $(\frac{1}{n},t_o) \notin p(B(x_o,\delta))$. This means any $(\frac{1}{n},t_o) \in C-\{(0,1)\}$ cannot be included in $p(B(x_o,\delta))$ or $p(B(x_o,\delta)) = \{(0,1)\} \subset B((1,0),\epsilon)$. Therefore, for any $x_o \in p^{-1}(\{(0,1)\})$, there is $B(x_o,\delta)$ such that $B(x_o,\delta) \subset p^{-1}p(B(x_o,\delta)) = p^{-1}\{(0,1)\}$, which implies $P^{-1}((0,1))$ is open. We can conclude $p^{-1}(0,1) = [0,1]$.

5. Let X be a connected and locally path connected space.

   (a) See Theorem 5.18.

   (b) See Theorem 5.19.