Refer to the Hilbert Axioms given below.

[AI 1] *For every point A and for every point B not equal to A, there exists a line a that passes through A and B.*

[AI 2] *For every point A and for every point B not equal to A, there exists no more than one that passes through A and B.*

[AI 3] *There exists at least two points on a line. There exists at least three points that do not lie on a line.*

[AI 4] *For any three points A, B, C that do not lie on the same line there exists a plane α that contains each of the points A, B, C. For every plane there exists a point which it contains.*

[AI 5] *For any three points A, B, C that do not lie on one and the same line there exists no more than one plane that contains each of the three points A, B, C.*

[AI 6] *If two points A, B of a line a lie in a plane α then every point of a lie in the plane α.*

[AI 7] *If two planes α, β have a point in common then they have at least one more point B in common.*

[AI 8] *There exist at least four distinct points which do not lie in a plane.*

[AO 1] *If $A * B * C$, then A, B, and C are three distinct points of a line, and $C * B * A$.*

[AO 2] *For two distinct points A and C, there always exists at least one point B lynig on the line $\overleftrightarrow{AC}$ such that $A * C * B$.*

[AO 3] *Of three distinct points lynig on the same line, there exists no more than one that lies between the other two.*

[AO 4] *Let A, B, and C be three points that do not lie on a line and let a be a line in the plane ABC which does not meet any of the points A, B, and C. If the line a passes through a point of the segment AB, it also passes through a point of the segment AC, or through a point of the segment BC.*

[AC 1] *If A and B are distinct points and if A' is any point, then it is always possible to find a point $B' \neq A'$ on any ray r emanating from A' such that segment AB is congruent or equal to the segment $A'B'$. In symbols,*

$$AB \cong A'B'$$

[AC 2] *If $CD \cong AB$ and $EF \cong AB$, then $CD \cong EF$,or briefly, if two segments are congruent to a third one they are congruent to each other.*

[AC 3] *If* $A * B * C$, $A' * B' * C'$, $AB \cong A'B'$ *and* $BC \cong B'C'$, *then* $AC \cong A'C'$.

[AC 4] *Given any* $\angle BAC$ *(where, by definition of "angle,"* $\overrightarrow{AB}$ *is not opposite to* $\overrightarrow{AC}$*), and given any ray* $\overrightarrow{A'B'}$ *emanating from a point* A', *then there is a unique ray* $\overrightarrow{A'C'}$ *on a given side of line* $\overleftrightarrow{A'B'}$ *such that* $\angle B'A'C' \cong \angle BAC$. *Moreover, every angle is congruent to itself.*

[AC 5] *If for two triangles* ABC *and* $A'B'C'$ *the congruences*

$$AB \cong A'B', AC \cong A'C', \angle BAC \cong \angle B'A'C'$$

holds, then the congruence

$$\angle ABC \cong \angle A'B'C'$$

is also satisfied.

[**Archimedes's Axiom**] *If* AB *and* CD *are any segments then there is a number* n *such that* n *segments* CD *costructed contiguously from* A, *along the ray from* A *through* B, *will pass beyond the point* B.

[**Line Completeness Axiom**] *An extension of a set of points on a line with its order and congruence relations that would preserve the relations existing among the original elements as well as the the fundamental properties of line order and congruence that follows from Axioms I - III and Archimedes's Axiom is impossible.*

[**Axiom of Parallelism**] *Let* α *be any plane,* a *be any line in* α, *and* A *a point in* α *not lying on* a. *Then there is at most one line in the plane,determined by* a *and* A, *that passes through* A *and does not intersect* a.

Midterm Examination for Introduction to Geometry (I) (Oct. 29, 2019)

1. In a geometric system satisfying [AI 1] through [AI 8], show that for every point there exists a plane through the point.

2. In a geometric system satisfying [AI 1] through [AO 4], for two points A and C, show that there always exists at least one point D on the line $\overleftrightarrow{AC}$ that lies between A and C.

3. In a geometric system satisfying [AI 1] through [AO 4], show that if $A * B * C$ and $B * C * D$, then $A * C * D$

4. In a geometric system satisfying [AI 1] through [AC 5],Show that right angles are congruent to each other.

5. In a geometric system satisfying [AI 1] through [AC 5], in a triangle $\triangle ABC$, show that $AB > BC$ if and only if $\angle BAC < \angle ACB$.

6. In a geometric system satisfying [AI 1] through [Line Completeness Axiom], show that if a geometric system Ω satisfies the axioms of incidence, order, congruence and continuity, then we cannot extend the system to a new system Ω^* with all the axioms of Ω preserved, by adding some new geometric elements such as points, lines or planes.

Solution of Midterm Examination for Introduction to Geometry (I) (Oct. 29, 2019)

1. Let A be a point in the system. Want to show there is a plane α containing A.

   1) There is another point B distinct from A. ([AI 3])

   2) There is a line l through A and B. ([AI 1])

   3)There is a point C which does not lie on l, otherwise all points lie on l, contradiction to [AI 3].

   4) Since A, B and C are not on the one and the same line, there is a plane α through those points. ([AI 4])

2. See Theorem 1.3.3.

3. Let g be the line where A, B, C, and D lie.

   1) Choose a point G that does not lie on g. ([AI 3]

   2) There exists a point F on a line $\overleftrightarrow{BG}$ with $B * G * F$. ([AO 2])

   3)The line $\overleftrightarrow{CF}$ meets neither the segment AB since $A * B * C$ nor the segment BG since $B * G * F$. (AI 2, AO 3)

   4) Note that $\overleftrightarrow{CF}$ does not pass through A, B, and G. The line $\overleftrightarrow{CF}$ does not meet the segment AG. (AO 4)

   5) Consider $\triangle BGD$. Note that $\overleftrightarrow{CF}$ does not pass through B, D, and G. Since $\overleftrightarrow{CF}$ meets the segment BD, and does not meet the segment BG, it must meet the segment GD at H with $G * H * D$. (AO 4)

   6) Consider the $\triangle AGD$. Since the line $\overleftrightarrow{CF}$ does not meet the segment AG, and meets the segment GD at H, it must meet the segment AD at C with $A * C * D$. (AO 3, AO 4)

4. See Theorem 1.3.26.

5. At first, let's show if $AB > BC$ then $\angle BAC < \angle ACB$.

   1) There is a unique $D \in \overrightarrow{BA}$ so that $BD \cong BC$, where $B * D * A$. ([AC 1] , Theorem 1.3.13)

2) Since $\overrightarrow{CD}$ is between $\overrightarrow{CA}$ and $\overrightarrow{CB}$, $\angle BCD < \angle BCA$. (def. 1.3.10)

3) Since $BD \cong BC$, $\angle BDC \cong \angle BCD$.

3) Since $\angle CDB$ is the exterior angle of $\triangle ACD$, $\angle BDC > \angle BAC$. (Theorem 1.3.27)

4) Since $\angle BAC < \angle BDC$, $\angle BDC \cong BCD$, and $\angle BCD < \angle ACB$, $\angle BAC < \angle ACB$. (Theorem 1.3.23, 25)

Conversely, let's show if $\angle BAC < \angle ACB$, then $AB > BC$.

1) If $AB \cong BC$, then $\angle BAC \cong \angle ACB$, a contradiction.

2) Suppose $AB < BC$.

3) There is a unique $D \in \overrightarrow{BC}$ so that $BD \cong BA$.([AC 1] , Theorem 1.3.13)

4) Since $BA \cong BD < BC$, $B * D * C$

5) Note that $\angle BAD \cong \angle BDA$. Since $B * D * C$, $\overrightarrow{AD}$ is between $\overrightarrow{AB}$ and $\overrightarrow{AC}$, and therefore $\angle BDA \cong \angle BAD < \angle BAC$.

6) Since $\angle BDA$ is an exterior angle of $\triangle ACD$, $\angle ACB = \angle ACD < \angle ADB \cong \angle BAD < \angle BAC$, a contradiction.

7) Therefore, $AB > BC$ by trichotomy.

6. See Theorem 1.3.29.