Final Examination for topology (Dec. 19, 2019)

1. Show that the unit n-cube

$$I^n = \{(x_1, \cdots, x_n) \in \mathbb{R}^n : \ 0 \le x_i \le 1 \text{ for } i = 1, \cdots, n\}$$

is a compact subspace of $\mathbb{R}^n$. (25 points)

2. Show that the one point compactification of the complex plane is homeomorphic to the unit sphere $\mathbb{S}^2$.(20 points)

3. Show that the product of a finite number of compact spaces is compact. (25 points)

4. Let $f : X \longrightarrow Y$ be a surjective continuous function and suppose that Y has the quotient topology determined by f. Prove that a function $g : Y \longrightarrow Z$ from Y to a space Z is continuous if and only if the composite function $gf : X \longrightarrow Z$ is continuous. (15 points)

5. Answer only one problem out of the following three problems. (15 points)

   (a) For the equivalence relation $(x, y) \sim (z, w)$ on $\mathbb{R} \times \mathbb{R}$ defined by $(x - z, y - w) \in \mathbb{Z} \times \mathbb{Z}$, is $\mathbb{R} \times \mathbb{R}/\sim$ homeomorphic to the torus $\mathbb{S}^1 \times \mathbb{S}^1$?

   (b) Let $f : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function defined by $f(x, y) = x + y$. Is the quotient space $\mathbb{R}^2/\sim_f$ homeomorphic to $\mathbb{R}$?

   (c) Let $ROT(O)$ be the group of rotations in $\mathbb{R}^2$ about the origin O. Define an action of $ROT(O)$ on the set $\mathbb{R}^2 - \{(0, 0)\}$ as $(A, (x, y)) \longmapsto A \begin{pmatrix} x \\ y \end{pmatrix}$ and an equivalence relation $\sim$ as $(x, y) \sim (z, w)$ if there is $A \in ROT(O)$ such that $A \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} z \\ w \end{pmatrix}$. Describe the quotient space $\mathbb{R}^2 - \{(0, 0)\}/\sim$.

Solution Set of Final Examination for topology (Dec. 19, 2019)

1. See the Lemma in p.172.

2. Consider the inverse stereographic projection $f : \mathbb{C}_\infty \longrightarrow \mathbb{S}^2$ defined by $f(z) = \left(\frac{z+\bar{z}}{|z|^2+1}, \frac{-i(z-\bar{z})}{|z|^2+1}, \frac{|z|^2-1}{|z|^2+1}\right)$ and $f(\infty) = (0,0,1)$. f is continuous at every $z \in \mathbb{C}$ because z, $\bar{z}$, $|z|^2+1$ are continuous and $|z|^2+1 \neq 0$. f is continuous at ∞ because for a basic open set $U = \{(x,y,z) \in \mathbb{S}^2 | x^2 + y^2 < r^2, z > 0\}$ of $f(\infty) = (0,0,1)$, $f^{-1}(U) = \{z \in \mathbb{C}_\infty | |z| > \frac{r}{1-\sqrt{1-r^2}}\}$ is open since its complement $\{z \in \mathbb{C}_\infty | |z| \leq \frac{r}{1-\sqrt{1-r^2}}\}$ is compact.

   Similarly we can show the stereographic projection $f^{-1} : \mathbb{S}^2 \longrightarrow \mathbb{C}_\infty$, defined by $(x,y,z) \mapsto \frac{x+iy}{1-z}$ and $(0,0,1) \mapsto \infty$, is continuous.

3. See Theorem 7.7 in p.202.

4. If $g : Y \longrightarrow Z$ and $f : X \longrightarrow Y$ are continuous, then $gf : X \longrightarrow Z$ is also continuous.

   Suppose $gf : X \longrightarrow Z$ is continuous. For an open set O in Z, $(gf)^{-1}(O) = f^{-1}(g^{-1}(O))$ is open in X. Since $f^{-1}(g^{-1}(O))$ is open in X and $g^{-1}(O) \subset Y$, $g^{-1}(O)$ is open in Y according to the definition of quotient topology determined by f. Therefore, g is continuous.

5. (a) Let $f : \mathbb{R}^2 \longrightarrow \mathbb{S}^1 \times \mathbb{S}^1$ be a continuous surjective function defined by

$$f(s,t) = ((\cos(2\pi s), \sin(2\pi s)), (\cos(2\pi t), \sin(2\pi t))).$$

   Notice that $(s,t) \sim (s',t')$ if and only if $f(s,t) = f(s',t')$ or $(s,t) \sim_f (s',t')$. Therefore, the quotient topology on $\mathbb{R} \times \mathbb{R}/\sim$ is in fact the quotient topology on $\mathbb{R}^2/\sim_f$. For a basic open set $B((s_o,t_o),\epsilon) = (s_o-\epsilon, s_o+\epsilon) \times (t_o-\epsilon, t_o+\epsilon)$ for sufficiently small $\epsilon > 0$,

$$\begin{aligned}
&f(B(s_o,t_o),\epsilon) \\
&= \{(\cos(2\pi s), \sin(2\pi s)) | s_o - \epsilon < s < s_o + \epsilon\} \\
&\times \{(\cos(2\pi t), \sin(2\pi t)) | t_o - \epsilon < t < t_o + \epsilon\},
\end{aligned}$$

   which is open in $\mathbb{S}^1 \times \mathbb{S}^2$. This means f is an open map. Therefore, $\mathbb{S}^1 \times \mathbb{S}^1$ has the quotient topology determined by f according to the Theorem 7.15. Let $h : \mathbb{R} \times \mathbb{R}/\sim \longrightarrow \mathbb{S}^1 \times \mathbb{S}^1$ be defined by $h([s,t]) = f(s,t)$ This is a homeomorphism because $\mathbb{S}^1 \times \mathbb{S}^1$ has the quotient topology determined by f, and Theorem 7.16.

(b) At first, let's think about the quotient topology determined by f and the usual topology. For a basic open interval (a,b) of the usual space $\mathbb{R}$, $f^{-1}((a,b)) = \{(x,y) \in \mathbb{R}^2 | a < x+y < b\}$, which is open infinite strip bounded by line $x+y=a$ and $x+y=b$. Hence any open interval (a,b) is also open in the quotient topology determined by f. Hence the usual topology of $\mathbb{R}$ is weaker than the quotient topology determined by f. Let O be a subset of $\mathbb{R}$ in which $f^{-1}(O)$ be open in $\mathbb{R}^2$. This means O is an open set in the quotient topology determined by f. Let $k \in O$. Then $f(0,k) = k \in O$ implies $(0,k) \in f^{-1}(O)$. Since $f^{-1}(O)$ is open in $\mathbb{R}^2$, we have $\{0\} \times (a,b) \subset f^{-1}(O)$ and therefore the open set $\{(x,y) \in \mathbb{R}^2 | a < x+y < b\} = f^{-1}((a,b))$ is contained in $f^{-1}(O)$. This means $k \in (a,b) \subset O$, and therefore O is the union of open intervals of $\mathbb{R}$. Therefore, the quotient topology determined by f is weaker than the usual topology of $\mathbb{R}$. Let $h : \mathbb{R}^2/\sim_f \longrightarrow \mathbb{R}$ be defined by h([k]) = k. Since $\mathbb{R}$ has the quotient topology determined by f, h is a homeomorphism by Theorem 7.16

(c) For a given non-zero vector (x,y), the equivalence class $[x,y]$ is a circle obtained by rotating (x,y) about the origin. Therefore, $\mathbb{R}^2 - \{(0,0)\}/\sim$ is in fact a deleted ray from the origin with the usual topology.